Why philosophy and maths walk hand in hand

Relational understanding in maths is about creating the links that make the subject exciting and revelatory. It shares a lot of principles with P4C, argues **Rod Cunningham**.

Evidently there is a bumper sticker on cars in California which says 'Philosophy is a game with objectives and no rules – mathematics is a game with rules and no objectives.'

If this is taken at face value it might be concluded that either philosophy and mathematics have nothing at all in common, or that they would usefully complement each other. This article attempts to show the latter, with particular reference to a practice known as Philosophy for Children (P4C). I will argue that good thinking and facilitation of thinking is similar in both a P4C community of enquiry and in a mathematics lesson designed to develop what Richard Skemp (1976) would call 'relational understanding'.

I will also argue that thinking and learning develops more effectively if these skills and strategies are made explicit and brought to the attention of both pupils and teachers, which a combination of P4C and exploratory mathematics sessions can do. I will attempt to give examples to demonstrate that cross-fertilisation of P4C and mathematics practice can broaden learning and teaching across the curriculum by providing mutual support for such skills.

Contexts for learning in mathematics

Richard Skemp (1976) refers to 'relational' and 'instrumental' understanding in mathematics. The first could be classed as deep understanding and is a function of the number of links made within the field of mathematics, the second an ability to follow a rule. An important part of achieving relational understanding is that learners' present thinking is challenged. John Mason (2004) refers to the Festinger idea of 'cognitive dissonance'.

'Deviations from expectation create disturbances which can lead to new sense-making. Without disturbance, there is no need to accommodate or assimilate' (Mason, 2004, p3). An example of an approach to mathematics learning with an emphasis on relational understanding is First Steps in Mathematics (FSiM), a diagnostic approach to the teaching of mathematics produced for the Western Australia Education Board.

This is based on a collection of international research into the difficulties children have in developing key mathematical concepts. A mantra of the FSiM approach is, 'You can get the right answer for the wrong reason.' Using FSiM diagnostic activities the underlying misconceptions held by developing mathematicians are uncovered and activities to overcome these are suggested. As FSiM shows, however, if conceptual misconceptions are not addressed they become compounded and preclude successful mathematical work outside direct application of learnt rules.

Seymour Papert claims that mathematical sense-making is best promoted within an exploratory framework. He sees children as 'builders of their own intellectual structures' who require 'the materials to build with'. Papert designed the programming language LOGO to allow exploration of the mathematical universe. This was underpinned by '..a view of learning mathematics in 'Mathland'; that is to say, in a context which is to learning mathematics what living in France is to learning French.' (Papert, 1980, p6). Researchers into the development of mathematical understanding using LOGO found, however, that whilst freedom for pupils to explore was critical, it was also important to provide some structure to bring them into the realm of key mathematical ideas. Hence the importance of the facilitator's role which is discussed next.

Examples: Promoting Relational Understanding

1) Let's Think through mathematics and CAME provide lessons based on collaborative enquiry methods and centred around key mathematical ideas. Details of courses and resources on: http://www.letsthink.org.uk

2) Marion Bird provides more open-ended mathematical starting points for posing questions and conducting enquiries.

3) Bowland Maths provides real-life problem-solving scenarios which develop skills of good thinking. http://www.bowlandmaths.org.uk

4) NRICH maths: Resouces for investigative mathematics
http://nrich.maths.org/frontpage

5) The First Steps in Mathematics scheme can provide guidance on key mathematical concepts and misconceptions. (it is now freely available on line)
http://det.wa.edu.au/stepsresources/detcms/navigation/first-steps-mathematics/

Facilitating relational learning in mathematics

According to Mason, achieving understanding of the relational kind in mathematics is not achieved by simply working collaboratively, doing and discussing mathematics. Mason refers to what he calls 'specializing' and 'generalising' (Mason 1988). He claims that meaning making in mathematics (and in other contexts) involves moving between particular cases and general statements, from one to the other, seeking out properties, exploring distinctions and relationships between cases, and using these as a basis for defining and reasoning.

The act of reasoning itself rests on a statement or conjecture about the situation, which arises from some puzzlement, some *cognitive dissonance*. According to Mason, the best way to develop these learning skills is to draw learners' attention to the processes, to bring them to the conscious attention of learners. For this to happen, of course, such moves must also be in the conscious attention of the teacher/facilitator. Such facilitation skills themselves need to be developed. This type of facilitation is the explicit aim of P4C training for teachers undertaken by SAPERE. P4C enquiries provide the opportunities to develop effective facilitation for good thinking and learning.

Philosophy for Children as a context for development of thinking and social skills

For over 40 years the practice of Philosophy for Children (P4C) has been applied in schools. A succinct description of P4C, its history and related research into impact can be accessed from the website of the charity SAPERE, which undertakes training and promotion in the United Kingdom. The emphasis in this article will be on the process of facilitating the Community of Enquiry (CoE) which is central to the practice. Experienced practitioners have described P4C as a 'thinking skills plus programme' which acknowledges both the cognitive and affective impact on pupils who have regular P4C sessions.

The training material for teacher/facilitators refers to the aim of developing creative, critical, collaborative and caring thinking. These are closely associated and intertwined rather than being discrete thinking strands. Enquiries focus on a philosophical question which has been devised and worked on by the participants and which concern an issue of importance to them. Dialogue is an essential part of a community of enquiry. Lev Vygotsky famously highlighted the importance of social interaction in the learning process, suggesting (in other words than these) that 'What the child can do today in community she can do tomorrow as an individual'. Susan Gardner suggests that this is not an automatic process but can happen as part of a well-facilitated enquiry.

Further Information: The website of the charity SAPERE, www.sapere.org.uk, provides a description of the practice, it's history, research into impact and links to training courses.

Enquiry in P4C as part of pushing for depth

The enquiry in P4C is far more than a straight-forward discussion. Susan Gardner's article, 'Inquiry is no Mere Conversation (or Discussion or Dialogue): Facilitation of Inquiry is hard work!', sums this up. 'The Community of Inquiry is neither teacher-centred and controlled nor student-centred and controlled, but centred on and controlled by the demands of truth.' (Gardner, 1995, p38).

To achieve this the facilitator needs the skills to be able to 'push for depth' in the dialogue, must be sensitive to philosophical ideas and thinking moves and must be able to maintain a degree of focus on the issues at hand. Over time, exposed to models of good practice, the pupils will themselves take on more of the facilitation role. The amount and timing of facilitator intervention is a delicate balance. Active involvement at a deep level and degree of ownership of the topics to be discussed by pupils is seen as a critical factor in developing their competence as thinkers and could be eroded if facilitators control too much.

In order to accommodate the requirements of pushing for depth and deepening discussion it is not unusual for enquiries to be undertaken over two, 50 minute sessions. The first session focusing on the formation and selection of an appropriate philosophical question, and the second on the discussion itself. This allows time for the facilitator, and the pupils themselves, to explore the issue, investigate philosophical links and devise some further varied activities around the concepts contained in the question.

P4C, Maths and facilitation in Practice

The challenge for mathematics educators, as if we didn't realise this already, is a complex one. How to provide the requisite context and freedom for exploration of key mathematical ideas including challenge to previous conceptions. This brings into sharp focus the role of the mathematics teacher as facilitator of learning. The term 'facilitation' is often used rather loosely to refer to 'a learning guide'. There is, however, a key lesson from the above discussions of facilitation in P4C and of deep learning in mathematics. This is that simple 'doing and discussing' (in the words of John Mason) is not enough in itself. Collaborative dialogue may be a vital ingredient but this needs to be accompanied by a 'push for depth' against criteria for good thinking.

The Good Thinker's Toolkit (Gregory, 2008) provides a practical guide to the elements of thinking that make up an effective Community of Enquiry. These are the staple of good P4C facilitation and can also be applied to maths lessons.

What?: asking for explanations, definitions, clarification of concepts and ideas.
Reasons: given or evaluated.
Assumptions: the importance of uncovering hidden assumptions.
Inferences: how does one idea follow from another?
Truth: Is a statement always/sometimes/never true? and how do we know?
Examples: found to support and exemplify a statement or argument.
Counter-examples: looked for in an attempt to undermine generalisations.

WRAITEC from (Gregory, 2008, p 48)

Examples:

1) **True or false?** Questions in mathematics push for conceptual clarification and, when included with questions of always/sometimes/never delineate the boundaries.

Is it true that multiplying a number always makes it bigger?
Is it true that shapes with longer perimeters have bigger areas?

Pupils who have discussed the question, 'Is it ever right to tell a lie? Will have experience in this type of exploration and use it across the curriculum including examples in mathematics.

2) **Uncovering assumptions:**
The following might be given as a reason during an enquiry,

'As a girl, you can't let on that you can do a maths problem 'cos you will be dropped by your friends'.

Pupils in P4C sessions will be used to exploring the assumptions behind this comment. In a mathematics lesson, when they evaluate statistical information they will be critical of how it was obtained and is being interpreted.

3) **Inference:** 'What if…?' Or 'What follows from…?' Type questions

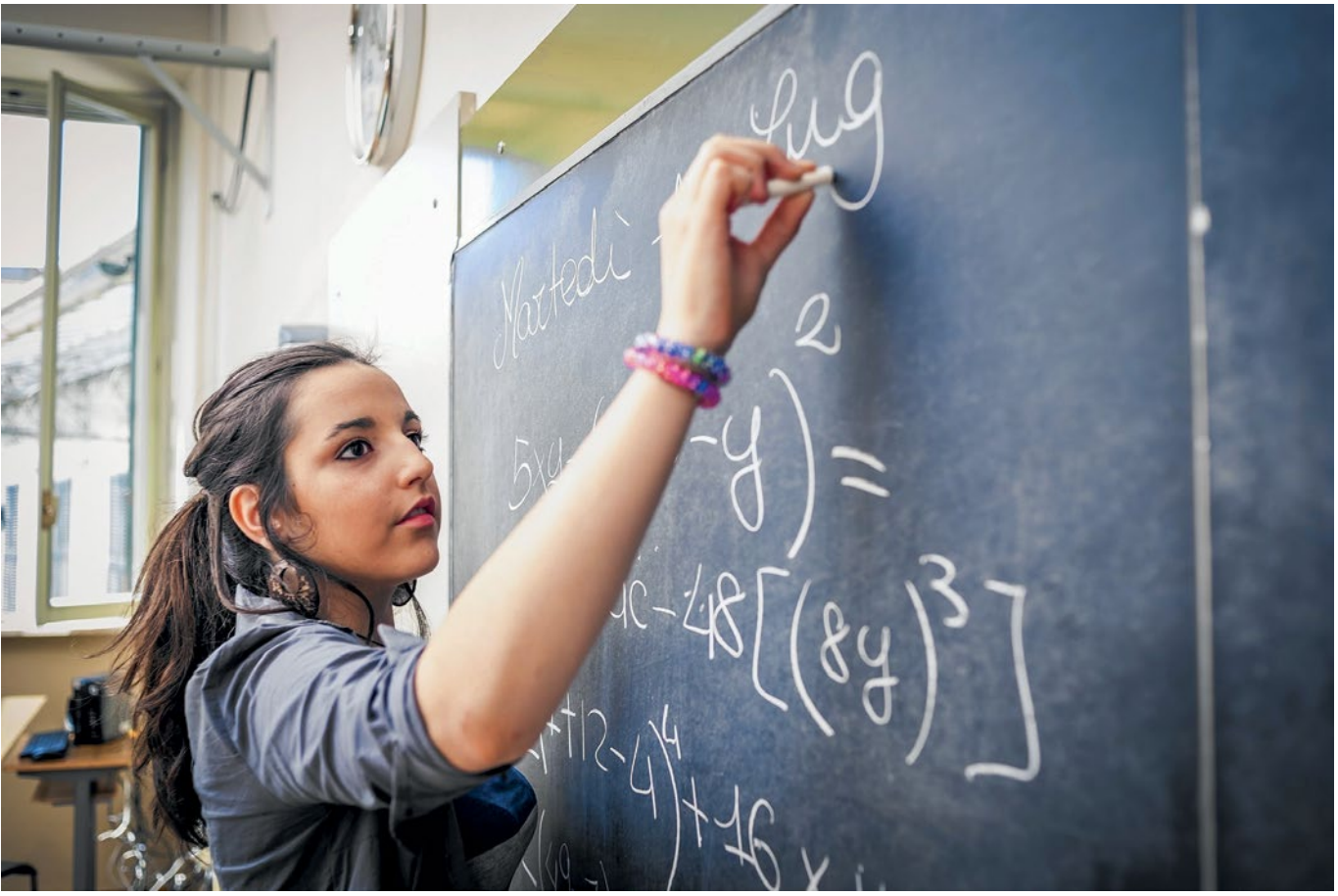
It might have been established that the angles in a triangle add up to 180 degrees. But then what happens if the triangle is drawn on the surface of a sphere, for example? What are the further implications for navigation across the globe? Readiness to ask such what if…? questions is a characteristic of P4C enquirers and valuable in developing relational understanding in all subjects.

4) **Induction and deduction:** In some cases, thinking developed in mathematics lessons can be applied in enquiries across the curriculum. A mathematical context provides good grounds for exploring the difference between inductive and deductive reasoning. Pythagoras' Theorem can be demonstrated visually and/or using logical constructions. The 100th member of a number sequence can be calculated using inductive methods or using analytic techniques. The appropriate use of examples and counter-examples becomes clear in the case of an inductive proof. On the other hand the generalised power of deduction can be demonstrated using a mathematical context. The understanding gained from such a comparison can then be applied in other settings, argumentation, scientific theory building et. cetera.

5) **Clarifying Concepts:** Mathematical concepts often seem abstract and far removed from every-day experience. A story or a comparison with a situation familiar to the learner can make such concepts and the relationships between them much clearer and provide useful 'hooks' for thinking and learning.

Example: Using a story to aid understanding of division for sharing further supported by a Let's Think Through Maths lesson. This example also features two other thinking skills, uncovering assumptions and questioning forming.

The Pat Hutchins' story, *The Doorbell Rang,* asks children to recalculate the number of cookies each person can have as more and more children arrive. This could introduce a series of lessons on sharing, to be followed up by 'How Many Each?' from the *Let's Think Through Maths 6-9* portfolio of lessons. As children will be aware, the issue of sharing in the real world is a complex one. In the Pat Hutchins' story we could ask, 'Should everyone get the same even if some children are much older?' There are assumptions embedded in *The Doorbell rang.* This is an appropriate time to demonstrate that hidden assumptions are worth revealing as part of good thinking. A further story *Four Feet, Two Sandals* relates a tale about two girls who each find one of a pair of sandals and have to decide who shall wear them. It ends with one girl deciding the other has the greater need and so should have them both. Division as sharing is as a tool for solving real problems but such problems require more than simple arithmetic processing.



Example: The story *Two Frogs*. By Chris Wormell explores the notion of risk through dialogue about the possibility of a particular set of circumstances occurring. This acts as a useful starter to a unit on probability leading on to further discussion of likelihood, and a better intuitive sense of the concepts involved. Younger pupils might then explore the use of probability language. An awareness of likelihood of an action's occurrence can be followed up by a explanation of how this can be quantified. In this vein, older students might use a spreadsheet to find out how many packs of football stickers one would have to buy to be 'sure' of obtaining the full set (do the organizers really hold some examples back in order to make it difficult or is this just a function of the mathematics of the situation?). This could lead to a more general discussion about perception of risk/probability and comparable mathematical formulations. More general would be the discussion of how mathematics can aid understanding in the face of conflicting intuitions.

Example: Using concept attainment approaches in mathematics and in relation to one's own experience in a community of enquiry.

'Friendship' is a concept which often arises in a P4C Community of Enquiry. For example the story book *Little Beauty* by Anthony Browne, led pupils to pose the question 'Do you have to be like someone to be a good friend to them? and developed into a discussion about what 'friendship' is. The facilitator devised some pen-pictures and scenarios which she brought to the next session and asked the pupils to sort them into 'definitely a friend' and 'definitely not a friend'. A few scenarios ended up in the disputed grey area in between. The exercise motivated the pupils to define key criteria for 'being a friend', albeit with some remaining disagreements. This highlighted the importance of defining concepts in both mathematics and in enquiries generally.

Example: Compare contrast, discriminate, borderline cases using inductive teaching and concept attainment.

A group of 2-D shapes or a group of numbers can be presented and the instruction given to 'sort according to criteria of your choice'. Alternatively an 'in-group' of numbers, shapes can be provided along with an 'out-group' and the challenge posed to 'find the criteria for being *in*'. These exercises can easily be adapted for any age group and provide useful practice in defining and making meaning whilst highlighting the possibility of variation.

6) **Questioning, Enquiring, Reasoning and Generalising:**

Example: Reasoning, question forming, enquiry and argumentation are key activities in both mathematics and P4C and can be usefully developed by working within both.

Pupils are not usually expected to pose questions for enquiry within a maths classroom. Seymour Papert's LOGO aside there are few opportunities to do so. P4C practice establishes the habit of question posing and exploration of different types of question. Enquiry of this sort can be beneficial in making links between concepts as well as developing the ability to persevere, provide reasons, defer judgement, listen to others and all the other aspects of good problem solving. Marion Bird provides a range of mathematical situations which are rich in problem-generating possibilities. Pupils experienced in P4C readily take to the sort of activity below:

Provide pupils with isometric dotty paper and ask them to draw a large triangle on it. Compare these in groups suggesting that groups come up with as many different types of triangle as they can (where the groups decide on criteria for 'being different'). Cut out the triangles and fold over each side in half to form perpendicular bisectors (this term may or may not be used at this stage). Paste the triangles on flip chart paper and continue the perpendicular bisectors with

ruler and pencil. Ask pupils in pairs to come up with questions about what they see. Some in the past have been 'Do they always meet at a point?' 'For which sorts of triangles do they meet inside?' 'Why do they meet on a side for some triangles?'

Facilitation should then press for summaries, hypothesis forming, testing, looking for counter-examples and reasons. Press further to ascertain if assertions are always true or only sometimes? Starting points can be found in all areas of mathematics. This way of working for at least part of the curriculum time establishes pupils as mathematical investigators rather than passive recipients of knowledge.

Further information: Marion Bird starting points can be found at: http://www.nationalstemcentre.org.uk/elibrary/maths/

Example: Forming generalisations from patterns, which is seen as difficult in mathematics, may be better understood within a philosophical enquiry. In an enquiry into 'bravery', for example, the process itself makes intuitive sense. Pupils can see that knowing about 'bravery' is important in the field of human relationships and that 'bravery' may take different forms within certain limits. They reach this conclusion by examining their own experience, listening to those of others and then identifying general features, perhaps testing out further particular cases against their own definition. This process then acts to exemplify mathematical generalisation, for example, about patterns in a 10 by 10 number square.

Combining P4C and Maths further in broader contexts

As well as the establishment of good thinking, there are other reasons for utilising the enquiry methods of P4C alongside and within a mathematics scheme of work. Such a combination can help establish the status of mathematics in schools as a subject which helps address relevant and live issues.

Using a mathematics starter to stimulate discussion about social issues

Example: *Design a Desk* is a PCAME activity available through the Let's think site (referenced below). It requires pupils in a class to take physical measurements and then design a desk which meets the needs of the majority of people in the class. The class might want to explore the idea of average, of difference and of equality of opportunity which could be stimulated by reading the story *Michael* by Tony Bradman. This relates the exploits of a school boy who is 'different' from his classmates. Past questions posed by pupils after reading this story include 'Is it OK to be different?' The combination of story and desk design activity leads to further discussion about different 'average' measures, their use and abuse.

Example: The book *If the World Were a Village* by David Smith is a graphic portrayal of world demographics using a village of 100 people. (for example, 22 out of the 100 people speak a Chinese dialect). Many other stimulating activities relating to social and environmental issues can be found in *Teaching Mathematics as if the Planet Matters.* By Coles et. al. These include 'The Fairtrade Game' and 'A Sustainable School'. The daily news provides numerous starting points for questioning and enquiry into environmental and social issues which makes the subject relevant and topical.

Example: Ways of seeing the world and describing it have been a pursuit of both mathematicians and philosophers. Having some idea of the history of both

and the puzzles and questions which are associated with this history is another important way of bringing mathematics to life. *The Librarian who Measured the Earth*. By Kathryn Lasky tells the story of measurement of the diameter of the earth using measurement of the sun's shadow. Pupils used to P4C enquiries will ask questions about how we build knowledge and the place of mathematics in this process.

Mathematics as a powerful communication and decision making tool.

Much decision-making utilises evidence couched in mathematical language and utilising mathematical processes. In order to enter into value judgements incorporating such information, citizens need to have a good grasp of both.

Example: Data displayed in different ways can be used to stimulate discussion about the use of statistics as part of the presentation of an argument. Starting with information about the salary structure of an international company, pupils might be tasked to present the information from different viewpoints, the higher paid in justification of the differentials and the lower paid requesting closer parity of monetary reward. This activity in itself will give rise to questions for enquiry about differentials in reward. 'Is inequality inevitable?' for example.

In purely instrumental terms, mathematical fluency opens doors to job opportunities. As a corollary to above much personal effectiveness and advancement in the twenty-first century rests on a facility with mathematical knowledge and skills. By introducing enquiry into units of mathematics the relevance of the subject becomes more apparent and the skills to use mathematical techniques as part of argument building are developed.

Aesthetics and mathematics

Questions about pattern, elegance, simplicity and complexity are fundamental to the process of making sense of the world. Philosophical enquiry questions the basis for establishing criteria and making judgements and helps equip pupils to become better at these important skills.

Example: A good starting point linking mathematics and art is to look at graphic representations of the Madelbrot set. Is this art? Are they beautiful? What about other patterns in nature? Does nature produce art? The Fibonacci sequence could also act as a stimulus for enquiry. In this case, some enquiry questions may be more factual than philosophical, for example, 'Why is the Fibonacci sequence observed in so many contexts?' This may lead to more philosophical questions about the relationship between mathematics and nature.

Example: Perspective is another area which lends itself to broader reflection and enquiry. Esher prints and optical illusions in general lead to enquiries about representation and modelling. The mathematical structure of Bach fugues and musical chords and dischords in general raises questions about the value we place on different sounds. In the cases above taking time to enquire enriches understanding and motivates learning.

Relational understanding is predicated upon engagement with the subject, with questioning and curiosity leading to enquiry and effective thinking and argumentation. The number of links that can be made within the subject and with the learner's experiences will, in large part, determine the ability of the learner to internalise, use and apply this knowledge. Incorporating P4C enquiry within the mathematics scheme of work builds capacity of both teachers and pupils to enjoy and succeed in the teaching and learning process.

Further information: More examples can be found in Cunningham and Smith (2012). Rod Cunningham is now a trainer for P4C and an ex mathematics teacher and adviser. He is collaborating with Nick Chandley to develop a one day P4C and mathematics course which can be booked through www.sapere.org.uk

Rod Cunningham is a registered trainer and trustee of the charity SAPERE which promotes Philosophy for Children in schools, colleges and communities in the UK. Having worked as a mathematics teacher and advisor for over 30 years, Rod now trains school staff to apply P4C to maths, science and whole-school practice. For more information about Rod's work with schools, visit his website: sustainable-enquiry.com.

Resources

Let's Think Resources
Let's Think Through Maths for ages 5 -6 and 6 – 9
Thinking maths in the Primary School PCAME
Available through GL Assessment
http://www.gl-assessment.co.uk/products/lets-think-through-maths
and information from the let's think website: http://www.letsthink.org.uk

Some Story Books to use with mathematics and P4C
Allen, P. 1980 *Mr Archimedes' Bath.* London: Puffin Books.
Browne, A. 2008 *Little Beauty.* London: Walker Books.
Carle, E. 1992 *The Very Hungry Caterpillar* London, Mantra Lingua.
Hutchins, P. 1986 *The Doorbell Rang* New York: Greenwillow Books.
Lasky, K. 1994 *The Librarian Who Measured the Earth.* New York: Little Brown
Lobell, A. 1971 *Frog and Toad Together.* New York: Harper Collins
Lord, J. 1972 *The Giant Jam Sandwich* London: Red Fox.
Pinczes, E. 1995 *A Remainder of One* New York: Houghton Mifflin Co.
Smith, D. J. 2004. *If the World Were a Village.* London: A and C Black Publishers.
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Bradman, T. 1990 *Michael.* New York: Macmillan

Pictures with a mathematical theme:
Pictures such as The Chambered Nautilus, Escher Prints, Hubble telescope images and the Mandelbrot Set can be used to stimulate enquiry.
http://www.seasky.org/deep-sea/chambered-nautilus.html
http://www.mcescher.com/
http://hubblesite.org/gallery/

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15. National Stem Centre, mathematics resources http://www.nationalstemcentre.org.uk/elibrary/maths/
16. Griffiths, P. Doing and Construing available at http://www.nationalstemcentre.org.uk/elibrary/maths/search?term=Doing+and+construing&filter=R&order=score accessed 09/11/15
17. NRICH is a mathematical website posing open-ended questions http://nrich.maths.org/public/
18. Bowland maths establishes problem solving scenarios for Key Stage 3 http://www.bowlandmaths.org.uk/